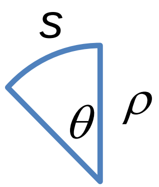
How Unit Vectors Change With Time

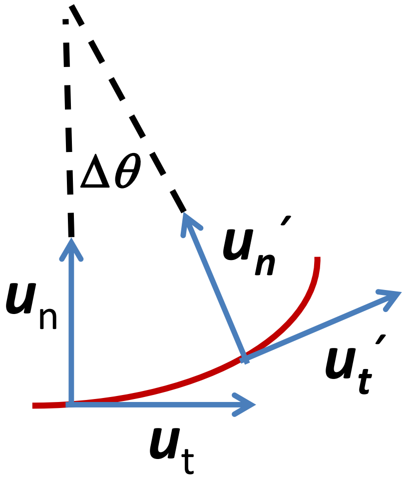
First note $\frac{d\hat{u}_t}{dt} = \frac{d\hat{u}_t}{d\theta}\frac{d\theta}{dt}$ using chain rule.

Next $\theta = \frac{s}{\rho}$, so $\frac{d\theta}{dt} = \frac{1}{\rho}\frac{ds}{dt} = \frac{v}{\rho}$

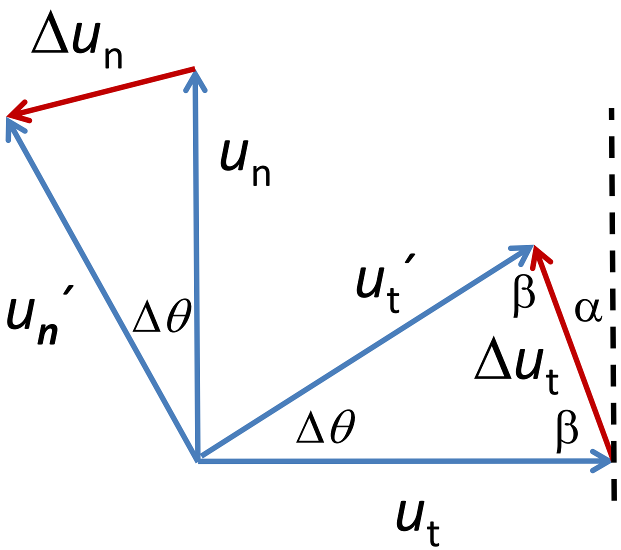
Need to determine $\frac{d\hat{u}_t}{d\theta}$



Consider how our unit vectors change as we move through some small angle $\Delta\theta$.



We will redraw unit vectors at common origin to examine $\Delta\theta$.



We have Isosceles Triangles since our unit vectors are all length 1.

$$2\beta + \Delta\theta = 180°$$

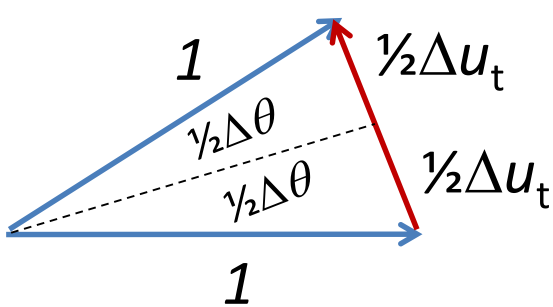
$$\beta + \alpha = 90°$$

$$\therefore \ \alpha = \frac{\Delta\theta}{2}$$

$$\overrightarrow{\Delta u_t} = \Delta u_t\left(-\hat{u}_t sin\left(\frac{\Delta\theta}{2}\right) + \hat{u}_n cos\left(\frac{\Delta\theta}{2}\right)\right)$$

$$\frac{\overrightarrow{\Delta u_t}}{\Delta\theta} = \frac{\Delta u_t}{\Delta\theta}\left(-\hat{u}_t sin\left(\frac{\Delta\theta}{2}\right) + \hat{u}_n cos\left(\frac{\Delta\theta}{2}\right)\right)$$

We have to find Δu_t to continue. Will break our Isosceles Triangle into two right triangles. We find



$$sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta ut}{2}$$

For small φ, $\sin\varphi \cong \varphi$ and $\cos\varphi \cong 1$

So $\frac{\Delta\theta}{2} = \frac{\Delta ut}{2}$ or $\Delta\theta = \Delta ut$

Thus $\frac{\overrightarrow{\Delta u_t}}{\Delta\theta} = \frac{\Delta u_t}{\Delta\theta}\left(-\hat{u}_t sin\left(\frac{\Delta\theta}{2}\right) + \hat{u}_n cos\left(\frac{\Delta\theta}{2}\right)\right)$ becomes

$$\frac{\overrightarrow{\Delta u_t}}{\Delta\theta} = -\hat{u}_t\frac{\Delta\theta}{2} + \hat{u}_n$$

We take the limit as $\Delta\theta \rightarrow 0$, to get the derivative $\frac{\overrightarrow{du_t}}{d\theta} = \hat{u}_n$

And thus $\frac{d\hat{u}_t}{dt} = \frac{d\hat{u}_t}{d\theta}\frac{d\theta}{dt} = \hat{u}_n\frac{v}{\rho}$

Using the same approach

$$\frac{d\hat{u}_n}{dt} = \frac{d\hat{u}_n}{d\theta}\frac{d\theta}{dt} = -\hat{u}_t\frac{v}{\rho}$$