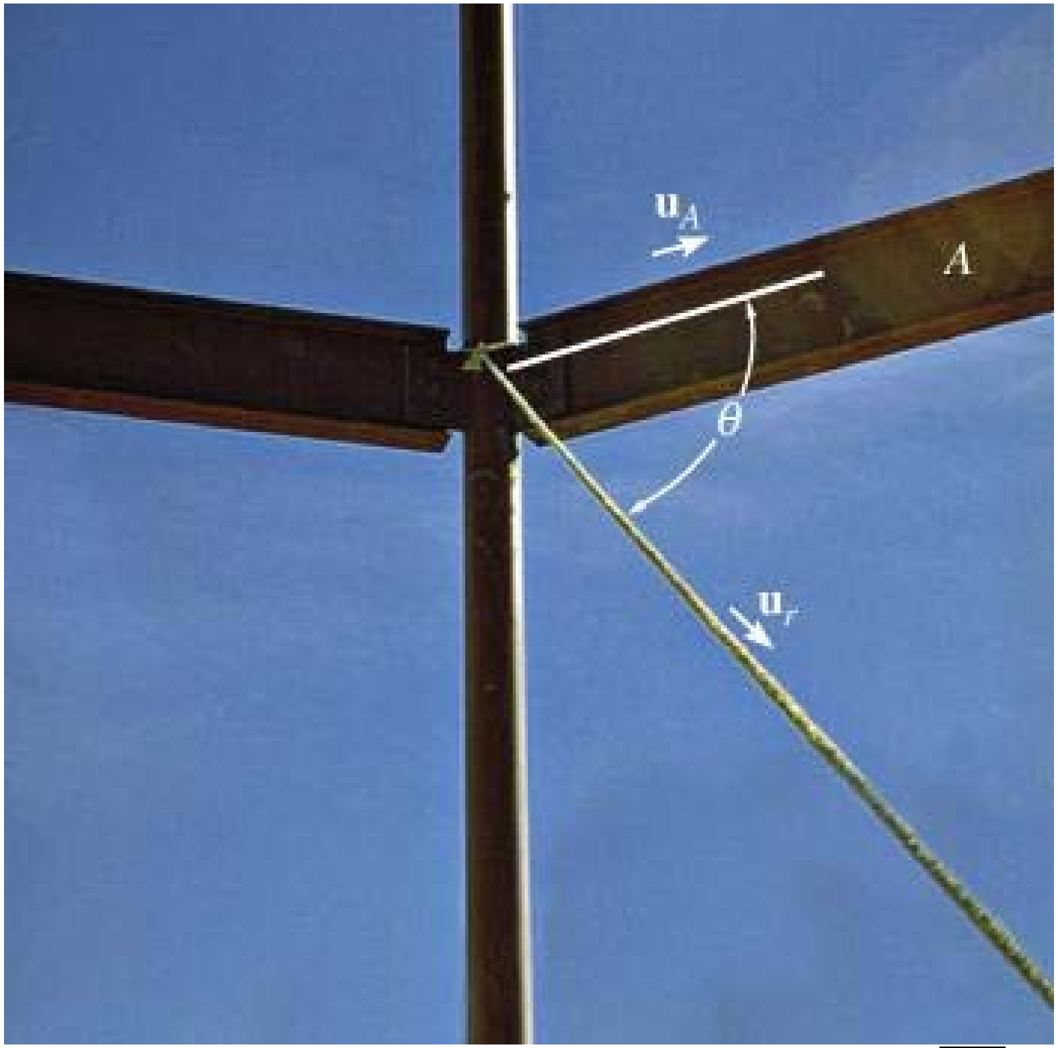
DOT PRODUCT

<u>Objective</u>:

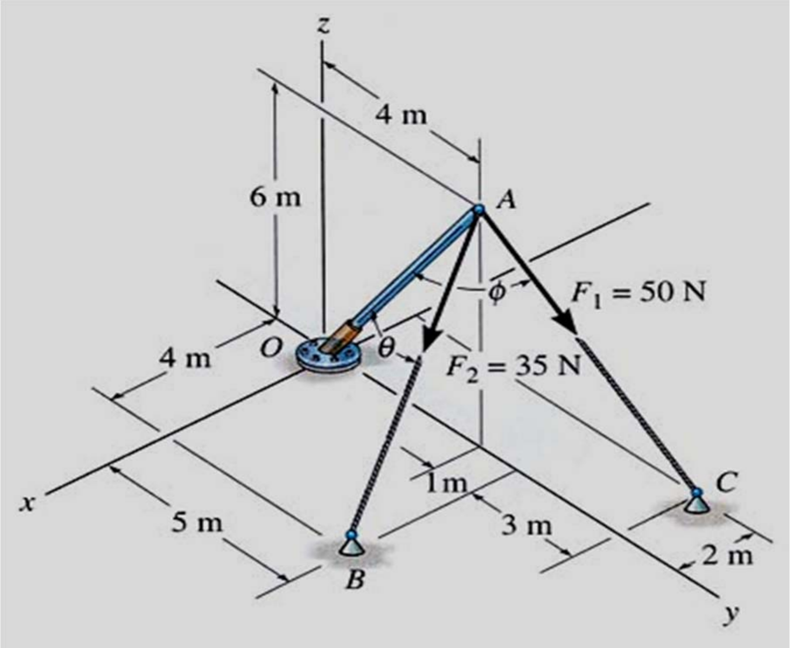
Students will be able to use the dot product to:

a) determine an angle between two vectors, and,

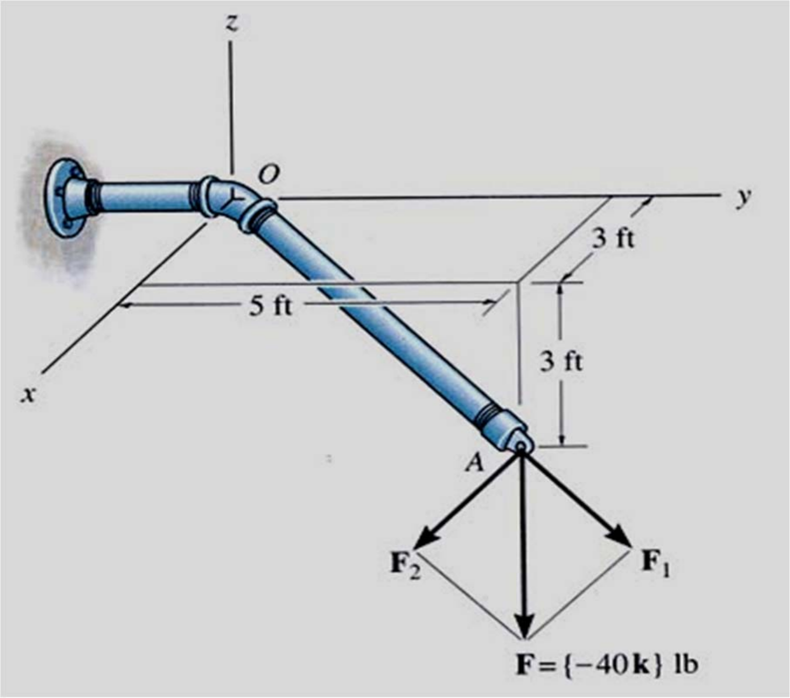
b) determine the projection of a vector along a specified line.



APPLICATIONS

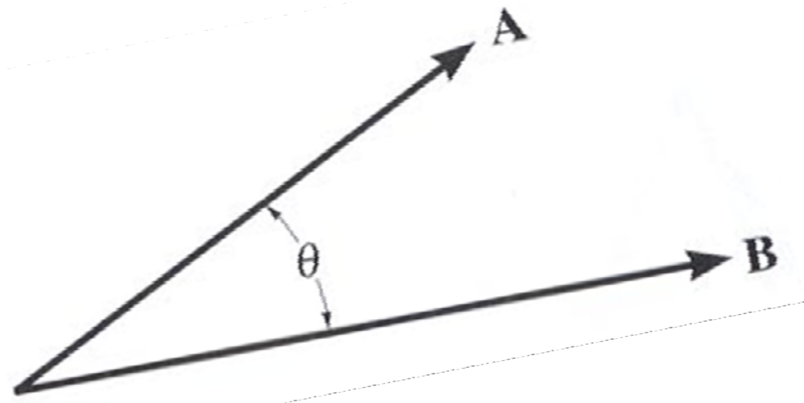


For this geometry, can you determine angles between the pole and the cables?



For force ***F*** at Point A, what component of it (F_1) acts along the pipe OA? What component (F_2) acts perpendicular to the pipe?

DEFINITION



The dot product of vectors $\vec{A}$ and $\vec{B}$ is defined as $\vec{A}\bullet\vec{B}$ = A B cos θ.

Angle θ is the smallest angle between the two vectors and is always in a range of 0° to 180°.

Dot Product Characteristics:

1. The result of the dot product is a scalar (a positive or negative number).
2. The units of the dot product will be the product of the units of the ***A*** and ***B*** vectors.

DOT PRODUCT DEFINITON

(continued)

The dot product, $\vec{A} \bullet \vec{B} = A\ B \cos\theta$, is easy to evaluate for $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

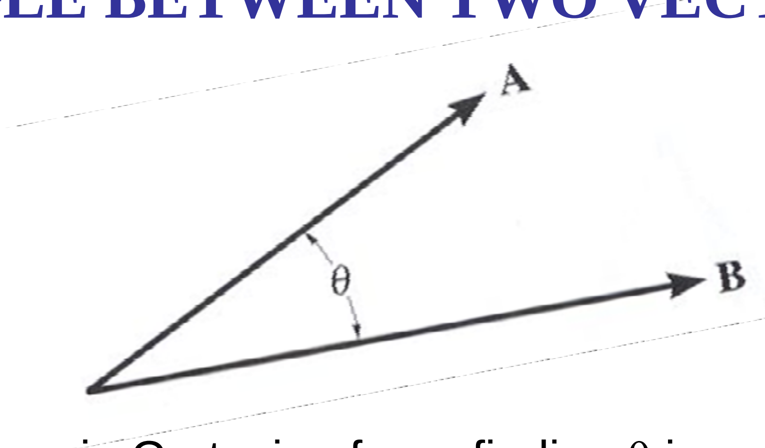
Examples: $\hat{\imath} \bullet \hat{\jmath} = 0$, $\hat{\imath} \bullet \hat{\imath} = 1$, and so on.

As a result, the dot product is easy to evaluate if you have vectors in Cartesian form.

$$\vec{A} \bullet \vec{B} = (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}) \bullet (B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k})$$

$$= A_x B_x + A_y B_y + A_z B_z$$

USING THE DOT PRODUCT TO DETERMINE THE ANGLE BETWEEN TWO VECTORS



If we know two vectors in Cartesian form, finding θ is easy since we have two methods of doing the dot product.

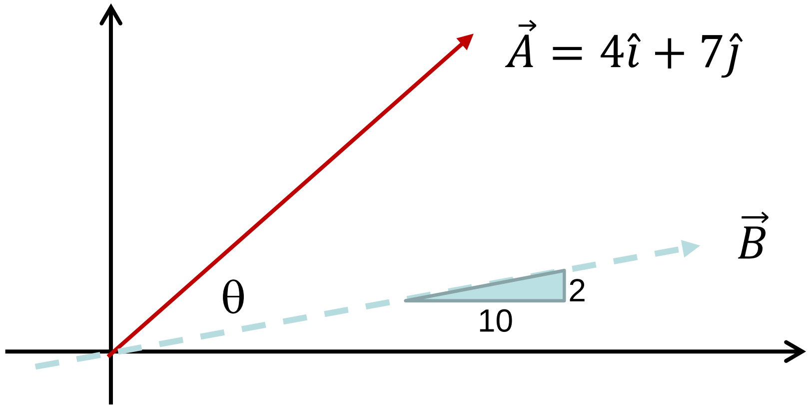
$$AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{A\ B}$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

More usually just written $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{A\ B}$ or $\cos\theta = \hat{u}_A \cdot \hat{u}_B$

2D Example – Find θ



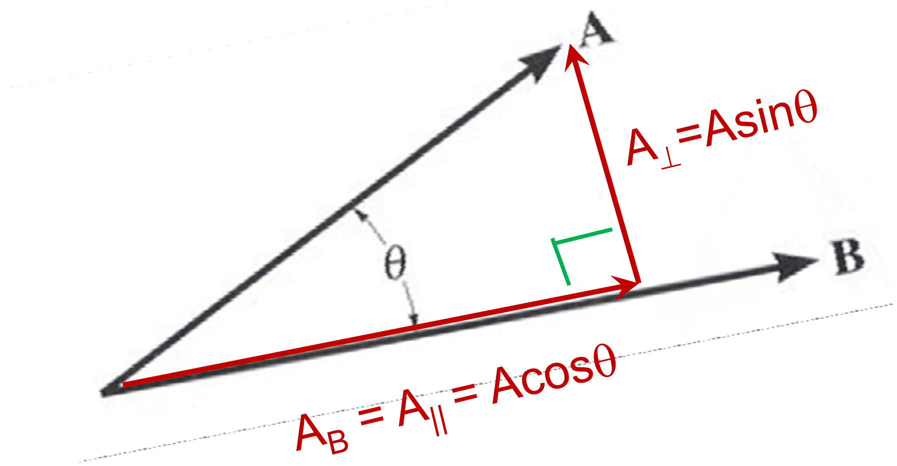
First $\hat{u}_B = \frac{10}{\sqrt{104}}\hat{\imath} + \frac{2}{\sqrt{104}}\hat{\jmath}$ and $\hat{u}_A = \frac{4}{\sqrt{65}}\hat{\imath} + \frac{7}{\sqrt{65}}\hat{\jmath}$

$$\begin{aligned}\cos\theta &= \hat{u}_A \cdot \hat{u}_B \\ &= \frac{4}{\sqrt{65}}\frac{10}{\sqrt{104}} + \frac{7}{\sqrt{65}}\frac{2}{\sqrt{104}} \\ &= \frac{54}{\sqrt{65}\sqrt{104}}\end{aligned}$$

So $\theta = 48.95°$

Projection

A dot product finds how much of $\vec{A}$ is in the same direction as $\vec{B}$ and then multiplies it by the magnitude of B



$$\vec{A} \cdot \vec{B} = A_B B$$

If we divide both sides in the of the definition above by the magnitude ***B***, we can get the magnitude of the *projection* of $\vec{A}$ in that direction.

$$A_B = \vec{A} \cdot \frac{\vec{B}}{B} = \vec{A} \cdot \hat{u}_B$$

Projection (cont)

It is easy to find $A_{\perp}$ as well, once we have A and A_B;

$$A_{\perp} = \sqrt{(A)^2 - (A_B)^2}$$

It is also easy to find $\vec{A}_B$. We already found the magnitude of the vector $\boldsymbol{A_B} = \vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{B}}$
And we know which way it points, $\hat{\boldsymbol{u}}_{\boldsymbol{B}}$. So

$$\overrightarrow{\boldsymbol{A}}_{\boldsymbol{B}} = \boldsymbol{A_B} \hat{\boldsymbol{u}}_{\boldsymbol{B}}$$

$$\overrightarrow{\boldsymbol{A}}_{\boldsymbol{B}} = \left(\overrightarrow{\boldsymbol{A}} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{B}}\right) \hat{\boldsymbol{u}}_{\boldsymbol{B}}$$

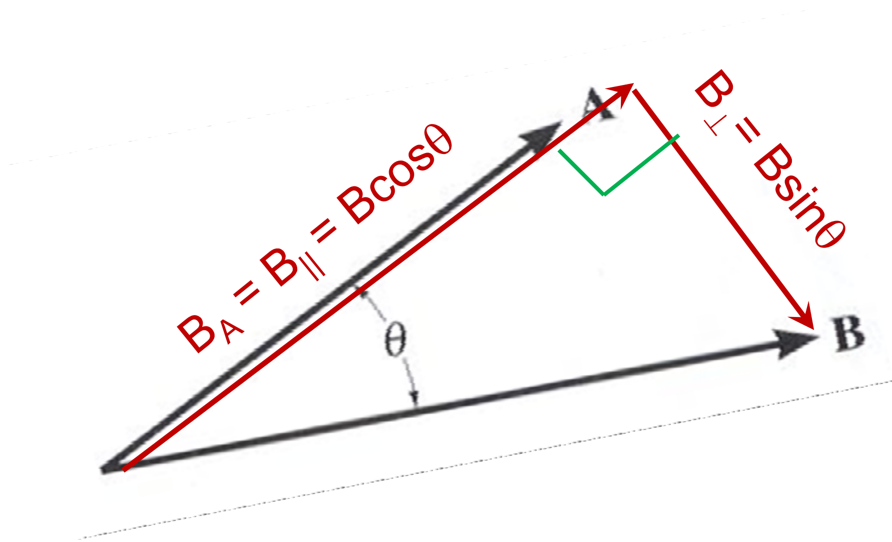
This looks a little odd because of the two unit vectors.

Once you have $\vec{\boldsymbol{A}}$ and $\vec{A}_B$, it is also easy to find $\vec{A}_{\perp}$.

$$\vec{A}_{\perp} = \vec{A} - \vec{A}_B$$

Projection (cont)

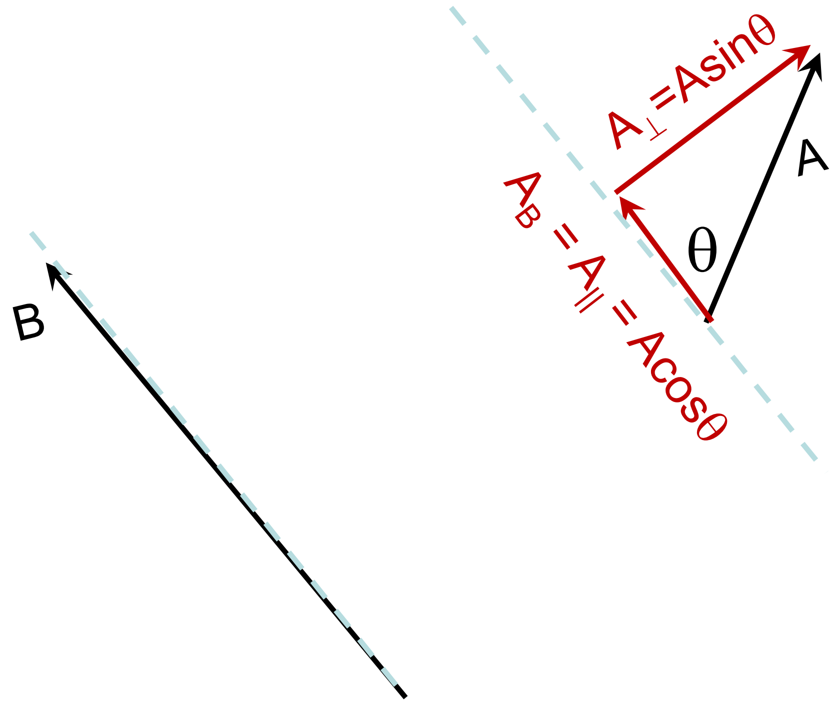
A dot product is commutative $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$



$$B_A = \vec{B} \cdot \frac{\vec{A}}{A} = \vec{B} \cdot \hat{u}_A$$

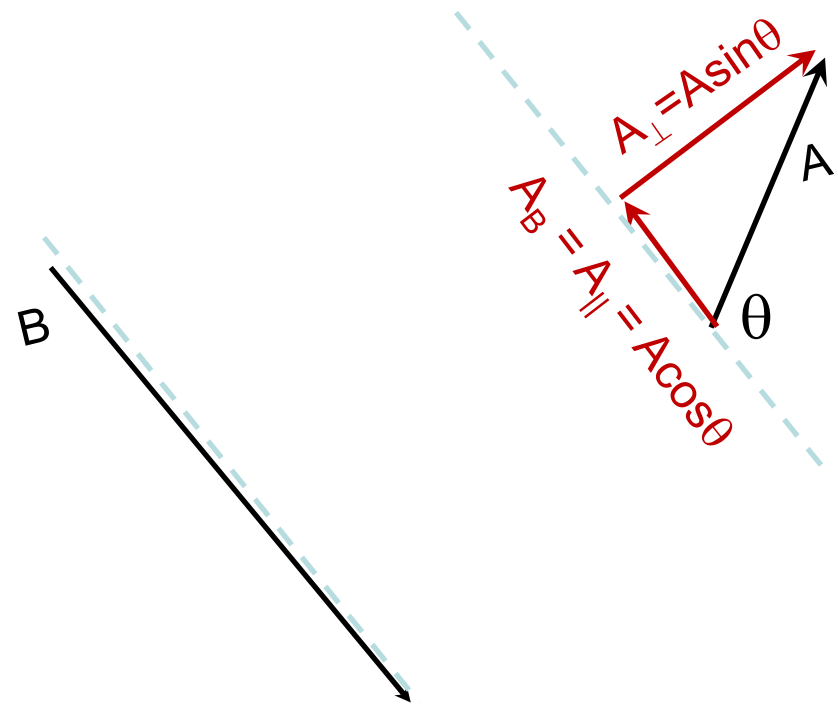
Note!

Vectors are not fixed to a point is space. Can do dot product on two vectors that are not touching and can find the angle between them and any projection of one vector along the other.

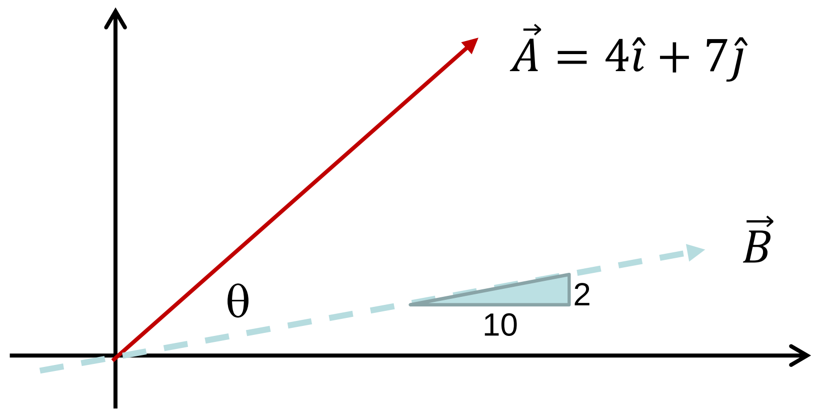


What if ***B*** is in the other direction?

Vectors are not fixed to a point is space. Can do dot product on two vectors that are not touching and can find the angle between them and any projection of one vector along the other.



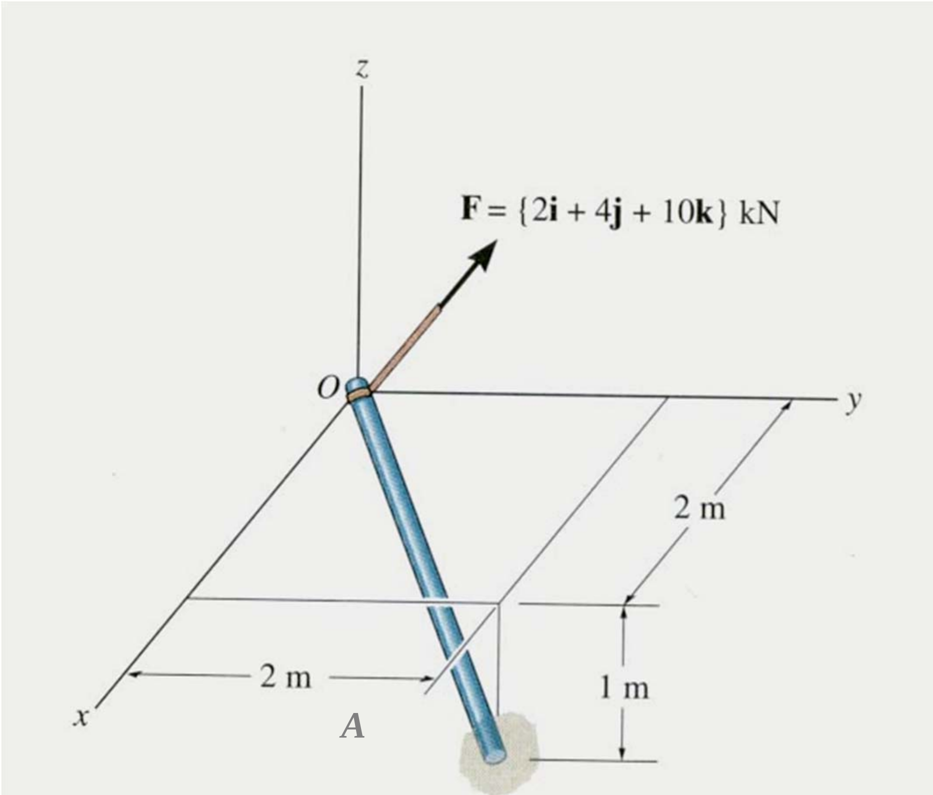
2D Example – Find $\vec{A}_{\text{B}}$ and $\vec{A}_{\perp}$



First $\hat{u}_B = \frac{10}{\sqrt{104}}\hat{\imath} + \frac{2}{\sqrt{104}}\hat{\jmath}$.

$$\vec{\boldsymbol{A}}_{\boldsymbol{B}} = \left(\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{B}}\right)\hat{\boldsymbol{u}}_{\boldsymbol{B}}$$
$$= \left\{(4\hat{\imath} + 7\hat{\jmath}) \cdot \left(\frac{10}{\sqrt{104}}\hat{\imath} + \frac{2}{\sqrt{104}}\hat{\jmath}\right)\right\}\left(\frac{10}{\sqrt{104}}\hat{\imath} + \frac{2}{\sqrt{104}}\hat{\jmath}\right)$$
$$= \left\{\frac{40}{\sqrt{104}} + \frac{14}{\sqrt{104}}\right\}\left(\frac{10}{\sqrt{104}}\hat{\imath} + \frac{2}{\sqrt{104}}\hat{\jmath}\right)$$
$$= \frac{540}{104}\hat{\imath} + \frac{108}{104}\hat{\jmath}$$

$$\vec{A}_{\perp} = \vec{A} - \vec{A}_B = (4\hat{\imath} + 7\hat{\jmath}) - \left(\frac{540}{104}\hat{\imath} + \frac{108}{104}\hat{\jmath}\right) = -\frac{124}{104}\hat{\imath} + \frac{624}{104}\hat{\jmath}$$



EXAMPLE

Given: The force acting on the pole

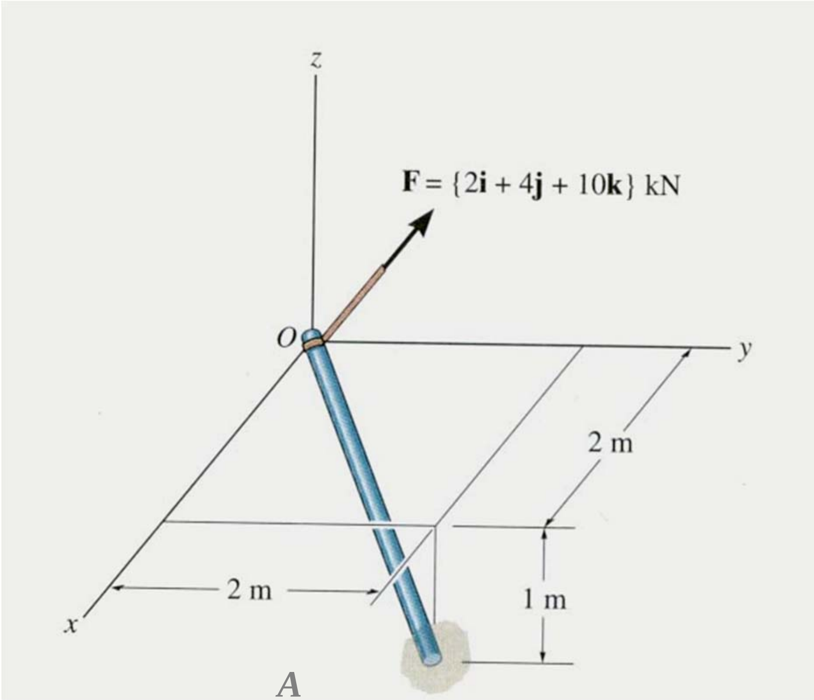
Find: The angle between the force vector and the pole, and the magnitude of the projection of the force along the pole OA.

Plan:

1. Get $\boldsymbol{r_{OA}}$

2. $\theta = \cos^{-1}\{(\boldsymbol{F} \bullet \boldsymbol{r_{OA}})/(F\ r_{OA})\}$

3. $F_{OA} = \boldsymbol{F} \bullet \boldsymbol{u_{OA}}$ or $F \cos\theta$



EXAMPLE
(continued)

$\boldsymbol{r_{OA}} = \{2\,\boldsymbol{i} + 2\,\boldsymbol{j} - 1\,\boldsymbol{k}\}$ m

$r_{OA} = (2^2 + 2^2 + 1^2)^{1/2} = 3$ m

$\boldsymbol{F} = \{2\,\boldsymbol{i} + 4\,\boldsymbol{j} + 10\,\boldsymbol{k}\}$kN

$F = (2^2 + 4^2 + 10^2)^{1/2} = 10.95$ kN

$\boldsymbol{F} \bullet \boldsymbol{r_{OA}} = (2)(2) + (4)(2) + (10)(-1) = 2$ kN·m

$\theta = \cos^{-1}\{(\boldsymbol{F} \bullet \boldsymbol{r_{OA}})/(F\, r_{OA})\}$

$\theta = \cos^{-1}\{2/(10.95 * 3)\} = 86.5°$

$\boldsymbol{u_{OA}} = \boldsymbol{r_{OA}}/r_{OA} = \{(2/3)\,\boldsymbol{i} + (2/3)\,\boldsymbol{j} - (1/3)\,\boldsymbol{k}\}$

$F_{OA} = \boldsymbol{F} \bullet \boldsymbol{u_{OA}} = (2)(2/3) + (4)(2/3) + (10)(-1/3) = 0.667$ kN

Or $F_{OA} = F\cos\theta = 10.95\cos(86.51°) = 0.667$ kN

CONCEPT QUIZ

1. If a dot product of two non-zero vectors is 0, then the two vectors must be _____________ to each other.

   A) parallel (pointing in the same direction)

   B) parallel (pointing in the opposite direction)

   C) perpendicular

   D) cannot be determined.

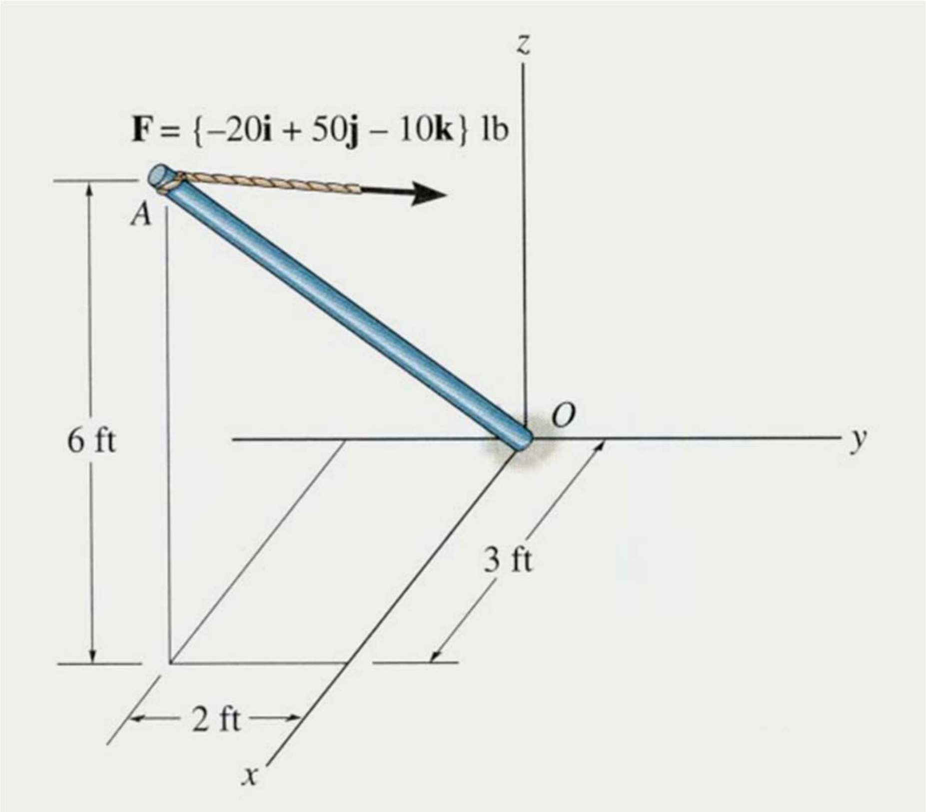
2. If a dot product of two non-zero vectors equals -1, then the vectors must be ________ to each other.

   A) parallel (pointing in the same direction)

   B) parallel (pointing in the opposite direction)

   C) perpendicular

   D) cannot be determined.



Example

Given: The force acting on the pole.

Find: The angle between the force vector and the pole, the magnitude of the projection of the force along the pole AO, as well as $\boldsymbol{F_{AO}}$ ($\boldsymbol{F}_{||}$) and $\boldsymbol{F}_{\perp}$.

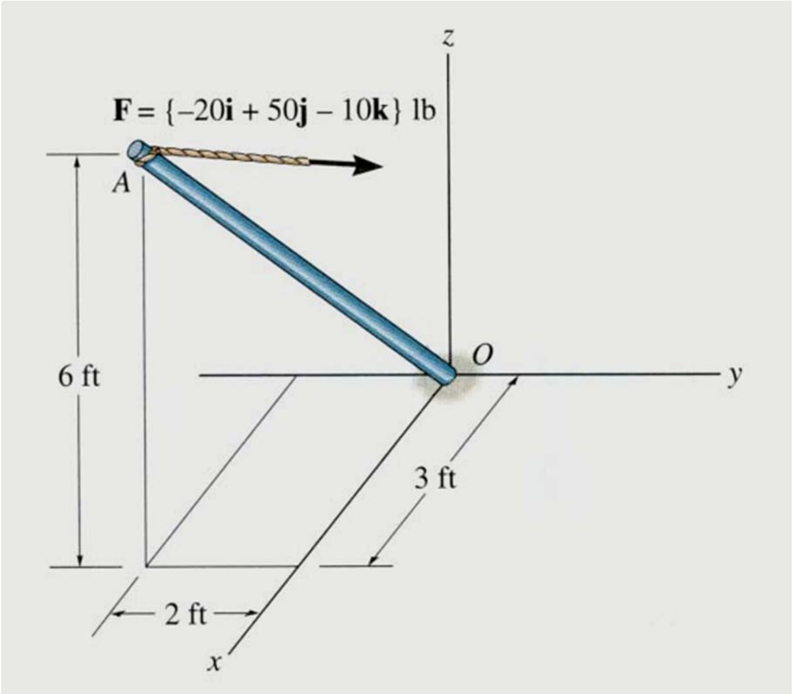
<u>Plan</u>:

1. Get $\boldsymbol{r_{AO}}$

2. $\theta = \cos^{-1}\{(\boldsymbol{F} \bullet \boldsymbol{r_{AO}})/(F\ \ r_{AO})\}$

3. $F_{AO} = \boldsymbol{F} \bullet \boldsymbol{u_{AO}}$ or $F \cos\theta$

4. $\boldsymbol{F_{AO}} = \boldsymbol{F}_{||} = F_{OA}\, \boldsymbol{u_{AO}}$ and $\boldsymbol{F}_{\perp} = \boldsymbol{F} - \boldsymbol{F}_{||}$



$\boldsymbol{r_{AO}} = \{-3\,\boldsymbol{i} + 2\,\boldsymbol{j} - 6\,\boldsymbol{k}\}$ ft.

$r_{AO} = (3^2 + 2^2 + 6^2)^{1/2} = 7$ ft.

$\boldsymbol{F} = \{-20\,\boldsymbol{i} + 50\,\boldsymbol{j} - 10\,\boldsymbol{k}\}$lb

$F = (20^2 + 50^2 + 10^2)^{1/2} = 54.77$ lb

$\boldsymbol{F} \bullet \boldsymbol{r_{AO}} = (-20)(-3) + (50)(2) + (-10)(-6) = 220$ lb·ft

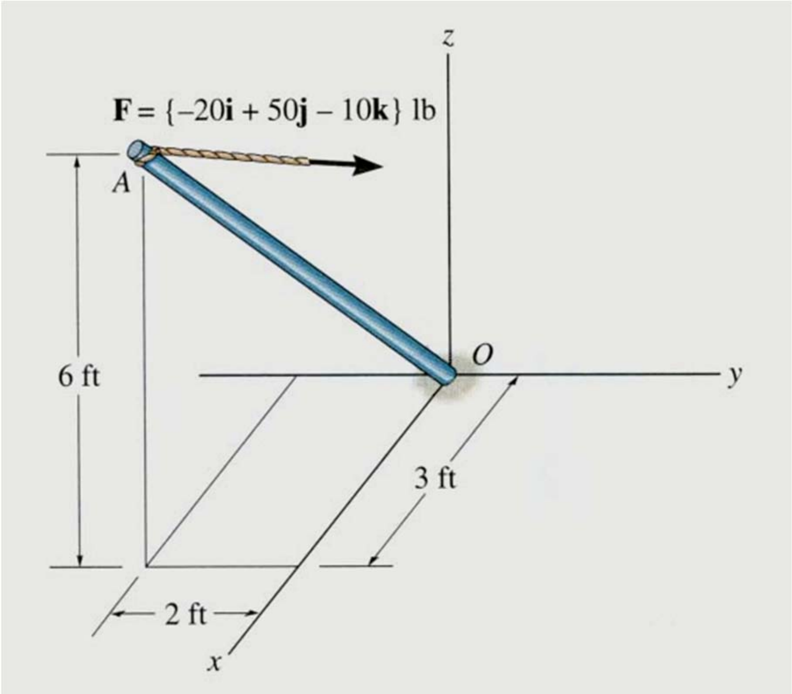
$\theta = \cos^{-1}\{(\boldsymbol{F} \bullet \boldsymbol{r_{AO}})/(F\, r_{AO})\}$

$\theta = \cos^{-1}\{220/(54.77 \times 7)\} = 55.0°$

$\boldsymbol{u_{AO}} = \boldsymbol{r_{AO}}/r_{AO} = \{(-3/7)\,\boldsymbol{i} + (2/7)\,\boldsymbol{j} - (6/7)\,\boldsymbol{k}\}$

$F_{AO} = \boldsymbol{F} \bullet \boldsymbol{u_{AO}} = (-20)(-3/7) + (50)(2/7) + (-10)(-6/7) = 31.4$ lb

Or $F_{AO} = F\cos\theta = 54.77\cos(55.0°) = 31.4$ lb



$$\boldsymbol{u}_{AO} = \{-3\,\boldsymbol{i} + 2\,\boldsymbol{j} - 6\,\boldsymbol{k}\}/7.$$

$$\boldsymbol{F}_{\parallel} = F_{AO}\,\boldsymbol{u}_{AO}$$

$$= (31.4\text{ lb})\,\boldsymbol{u}_{AO}$$

$$= \{-13.46\,\boldsymbol{i} + 8.97\,\boldsymbol{j} - 44.86\boldsymbol{k}\}\text{lb}$$

$$\boldsymbol{F}_{\perp} = \boldsymbol{F} - \boldsymbol{F}_{\parallel}$$

$$= \{-20\,\boldsymbol{i} + 50\,\boldsymbol{j} - 10\,\boldsymbol{k}\}\text{lb}$$

$$- \{-13.46\,\boldsymbol{i} + 8.97\,\boldsymbol{j} - 44.86\boldsymbol{k}\}\text{lb}$$

$$= \{-6.54\,\boldsymbol{i} + 41.03\,\boldsymbol{j} + 34.86\,\boldsymbol{k}\}\text{lb}$$

QUIZ

1. The Dot product can be used to find all of the following except _____ .

A) sum of two vectors

B) angle between two vectors

C) component of a vector parallel to another line

D) component of a vector perpendicular to another line

2. Find the dot product of the two vectors ***P*** and ***Q*.**

P = {5 ***i*** + 2 ***j*** + 3 ***k***} m

Q = {-2 ***i*** + 5 ***j*** + 4 ***k***} m

A) -12 m B) 12 m C) 12 m^2

D) -12 m^2 E) 10 m^2