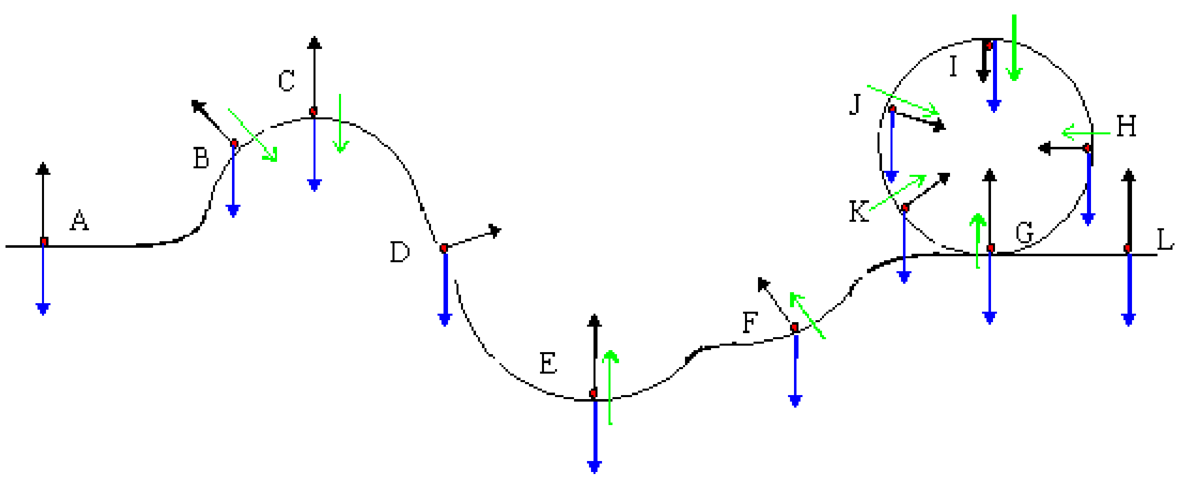
Physics 1100: Uniform Circular Motion & Gravity

1. In the diagram below, an object travels over a hill, down a valley, and around a loop-the-loop at constant speed v. At each of the specified points draw a free body diagram indicating the directions of the normal force and of the weight. Also indicate the magnitude and direction of the centripetal acceleration, if any. Write down the equations that Newton's Second Law gives you for the points with centripetal acceleration.

   Recall that weight always acts down which, by convention, is taken to be the bottom of the page. Normal forces act normal to the surface, from the surface through the object. To have a centripetal acceleration, an object must be travelling partially or wholly in a circle. Since points A, D, and L are not curved there is no centripetal acceleration at these points. For the rest of the points, the direction of the centripetal acceleration is by definition towards the centre of the curve. There may also be a component of acceleration with is tangential to the curves but we are not asked about that.

   In the diagram, normal forces are represented by black (dark) arrow, weight by blue arrows, and centripetal acceleration by green (grey) arrows.



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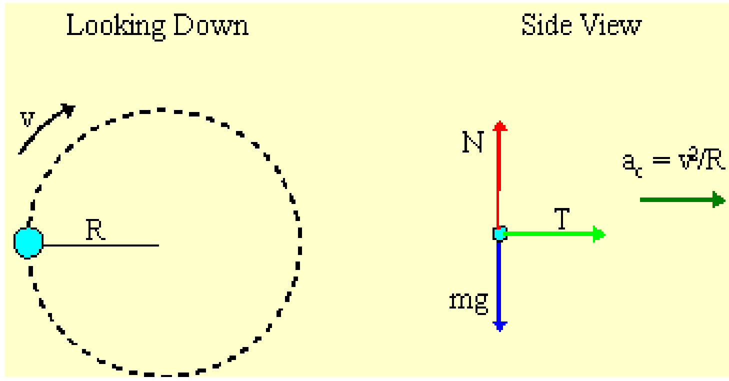
2. A 1.50-kg rock is being twirled in a circle on a frictionless surface using a horizontal rope. The radius of the circle is 2.00 m and the rope make 100 revolutions in 1.00 minutes. What is the tension in the rope.

   Since the rock is moving in a circle, it has an inward, centripetal, acceleration. We are asked to find a force, the tension. This suggests that we should draw a free-body diagram (FBD) and use Newton's Second Law. First we draw sketch of the motion, from the top and from the side as is useful in centripetal acceleration problems.



i	***j***
$F_x = ma_x$	$F_y = ma_y$
$T = mv^2/R$	$N - mg = 0$

The equation in the first column will tell us what the tension is, if we can determine v. For uniform motion in a circle,

$$v = d/t = 100 \text{ rev/min} = 100 \text{ rev} / 1 \text{ min} \times (2\pi(2.00 \text{ m})/\text{rev}) \times (1 \text{ min})/(60 \text{ s}) = 20.94 \text{ m/s} .$$

Thus

$$T = mv^2/R = (1.50 \text{ kg})(20.94 \text{ m/s})^2 /(2.00 \text{ m}) = 329 \text{ N} .$$

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3. The rope in question 2 will break when the tension exceeds 1000 N. What will be the speed of the rock just as the rope breaks?

   Now we have T and want v, so rearranging yields

   $$v = [FR/m]^{1/2} = [(1000 \text{ N})(2.00 \text{ m/s})/(1.50 \text{ kg})]^{1/2} = 36.5 \text{ m/s} .$$

   Converting to rev/min,

   $$36.5 \text{ m/s} = 36.5 \text{ m/s } (1 \text{ rev})/(2\pi(2.00\text{m})) (60 \text{ s} / 1 \text{ min}) = 174 \text{ rev/min} .$$

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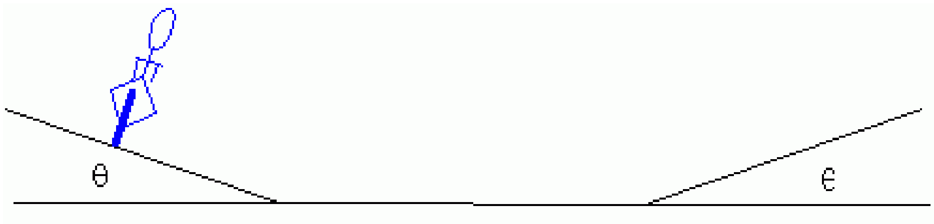
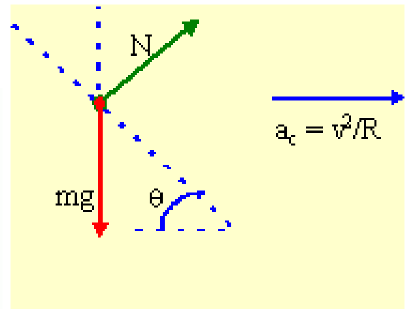
4. Bicycle racetracks, or velodromes, are banked at the ends. If we ignore friction, and the banking angle is θ, what will the maximum speed of a bicycle if it is to move around the end of the track at constant radius r? If the bicyclist goes faster that this value what will happen? If the bicyclist goes slower?

   Notice that the cyclist travelling in part of a circle, that suggests a centripetal acceleration. By definition, the centripetal acceleration points towards the centre of the velodrome; in this case to the right. Since this problem deals with centripetal acceleration, that suggests we should apply Newton's Second Law. To apply Newton's Second Law we draw a free-body diagram (FBD) for the cyclist. The cyclist has mass, so he has weight. There is the normal force from the incline on the cyclist.

Also we should choose a coordinate system where one axis point in the direction of the acceleration.





i	***j***
$F_x = ma_c$	$F_y = ma_y$
$N\sin(\theta) = mv^2/R$	$N\cos(\theta) - mg = 0$

We can rewrite the equation in the second column as $N = mg/\cos(\theta)$ and substitute this into the equation in the first column,

$$mg\sin(\theta)/\cos(\theta) = mv^2/R .$$

Now we can eliminate m, replace $\sin(\theta)/\cos(\theta)$ by $\tan(\theta)$, and solve for v to get

$$v = [gR\tan(\theta)]^{1/2} .$$

Examining our equation for v, we see that if v increases then R must increase if is a constant. The bicyclist can only increase R by moving up the incline. Similarly, if v decreases, the bicyclist must move down the incline.

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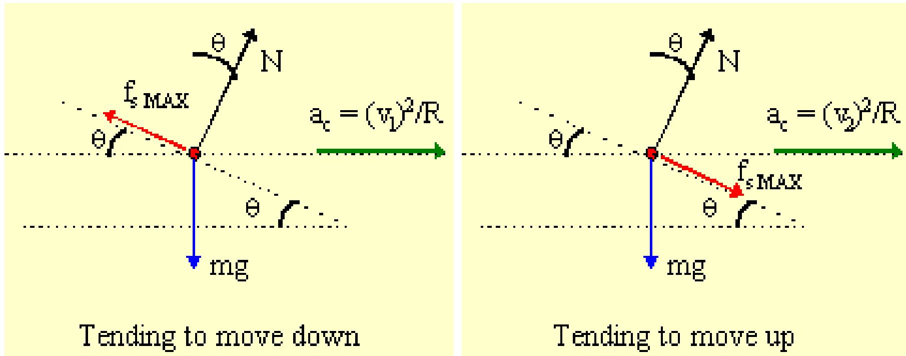
5. Suppose we do not ignore friction in question 4 and that the coefficient of static friction is μs. How fast can the bicyclist travel and still remain at radius r? How slow could the bicyclist ride and still travel in a circle?

   We are told that there is friction but it is not immediately clear which way it points. The fact that the problem asks for a range of velocities suggests that we are to solve the problem both ways, one with friction up the incline and one with it down.

   Keep in mind that even though the bicycle wheels are moving, they are not slipping, so it is static friction which applies. As well, experience and our results for question 4 tell us that if the cyclist moves too fast, he will tend to move up the incline. In such a case friction would resist this tendency an point down the incline. Similarly if the cyclist slows down, friction will be up the incline.



down

i	***j***
$F_x = ma_x$	$F_y = ma_y$
$N\sin(\theta) - f_{s\ MAX}\cos(\theta) = m(v_1)^2/R$	$N\cos(\theta) + f_{s\ MAX}\sin(\theta) - mg = 0$

We also know that $f_{s\ MAX} = \mu_s N$. If we substitute this into our two equations above, we find

$$N\sin(\theta) - \mu_s N\cos(\theta) = m(v_1)^2/R \text{, and}$$

$$N\cos(\theta) + \mu_s N\sin(\theta) = mg.$$

We have two equations in two unknowns, N and v_1. The second equation can be rewritten as $N = mg / [\cos(\theta) + \mu_s\sin(\theta)]$. Using this in the first equation and rearranging yields:

$$v_1 = \sqrt{gR\left(\frac{\sin(\theta) - \mu_s\cos(\theta)}{\cos(\theta) + \mu_s\sin(\theta)}\right)}$$

up

i	***j***
$F_x = ma_x$	$F_y = ma_y$
$N\sin(\theta) + f_{s\ MAX}\cos(\theta) = m(v_2)^2/R$	$N\cos(\theta) - f_{s\ MAX}\sin(\theta) - mg = 0$

We also know that $f_{s\ MAX} = \mu_s N$. If we substitute this into our two equations above, we find

$$N\sin(\theta) + \mu_s N\cos(\theta) = m(v_2)^2/R \text{, and}$$

$$N\cos(\theta) - \mu_s N\sin(\theta) = mg.$$

We have two equations in two unknowns, N and v_2. The second equation can be rewritten as $N = mg / [\cos(\theta) - \mu_s\sin(\theta)]$. Using this in the first equation and rearranging yields:

$$v_2 = \sqrt{gR\left(\frac{\sin(\theta) + \mu_s \cos(\theta)}{\cos(\theta) - \mu_s \sin(\theta)}\right)}$$

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6. The moon circles the earth once every 27.3 days. We have already determined that the mass of the earth is 5.98×10^{24} kg. What is the distance from the center of the earth to the centre of the moon?

   From Kepler's Law, we know

   $$T = 2\pi R^{3/2} / [GM_{central}]^{½} .$$

   If we rearrange this to find R, we get

   $$R = [GM_{central}T^2 / 4\pi^2]^{1/3} .$$

   Next we convert the period into seconds

   $$T = (27.3\ d) \times (24\ h) / (1\ d) \times (3600\ s) / (1\ h) = 2.3587 \times 10^6\ s .$$

   So we find the centre-to-centre distance radius to be

   $$R = [(6.672\times10^{-11}\ N\text{-}m^2/kg^2)(5.98\times10^{24}\ kg)(2.3587\times10^6\ s)^2 / 4\pi^2]^{1/3} = 3.83 \times 10^8\ m .$$

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7. The earth is a satellite of the Sun. The distance from the sun to the earth is 1.50×10^{11} m. What is the mass of the Sun?

   From Kepler's Law, we know

   $$T = 2\pi R^{3/2} / [GM_{central}]^{½} .$$

   If we rearrange this to find the mass of the Sun, we get

   $$M_{sun} = 4\pi^2 R^3 / GT^2 .$$

   Next we convert the period into seconds

   $$T = (365\ d) \times (24\ h) / (1\ d) \times (3600\ s) / (1\ h) = 3.154 \times 10^6\ s .$$

   So we find the Sun's mass to be

   $$M_{sun} = 4\pi^2(1.50\times10^{11}\ m)^3 / (6.672\times10^{-11}\ N\text{-}m^2/kg^2)(3.154\times10^6\ s)^2 = 2.01 \times 10^{30}\ kg .$$

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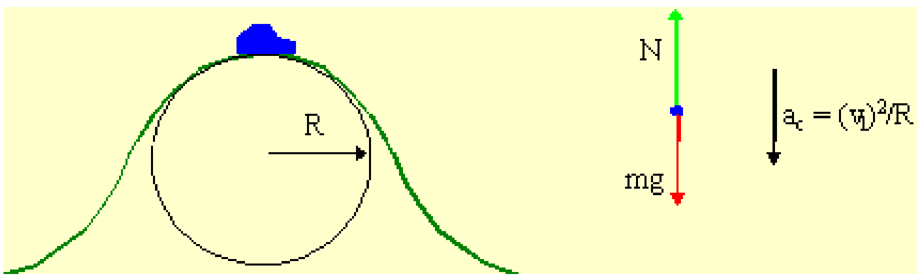
8. What is the apparent weight of a 75.0-kg person travelling at 100 km/h (a) over the peak of a hill with radius of curvature equal to 500 m, and (b) at the bottom of a hollow of the same radius?

Whenever we encounter a problem involving motion in a circle or part of a circle, we probably dealing with centripetal motion. Since we may assume the problem involves an acceleration and since a force, the apparent weight, is mentioned that suggests we should apply Newton's Second Law. To apply Newton's Second Law we draw a free-body diagram (FBD) for the person in each situation. The person has mass, so he has weight. There is the normal force from the surface acting on the person. The normal force is, of course, the apparent weight.

Centripetal acceleration acts towards the centre of the hill, i.e. downwards.



$$F_y = ma_y$$

$$N - mg = -m(v_1)^2/R$$

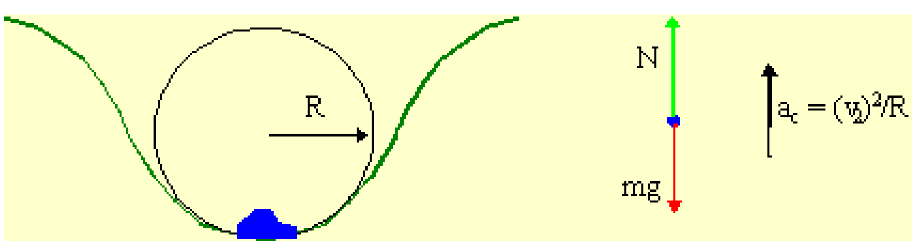
The apparent weight is thus,

$$N = mg - m(v_1)^2/R = (75.0\ \text{kg})[9.81\ \text{m/s}^2 - (27.778\ \text{m/s})^2/(500\text{m})] = 620\ \text{N}\ ,$$

where I have used the conversion

$$100\ \text{km/h} = 100\ \text{km/h}\ \times (1000\ \text{m})/(1\ \text{km}) \times (1\ \text{h})/(3600\ \text{s}) = 27.778\ \text{m/s}\ .$$

Centripetal acceleration acts towards the centre of the hill, i.e. downwards.



$$F_y = ma_y$$

$$N - mg = +m(v_1)^2/R$$

The apparent weight is thus,

$$N = mg + m(v_2)^2/R = (75.0\text{ kg})[9.81\text{ m/s}^2 + (27.778\text{ m/s})^2/(500\text{m})] = 851\text{ N} .$$

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9. What is the minimum speed for a rollercoaster to be remain in contact with the tracks if it is doing an upside down loop of radius 350 m.

Notice that the rollercoaster car is travelling in part of a circle, that suggests a centripetal acceleration. For the car to leave the track, the normal force between the car and the track must be zero. Since this problem deals with a force, the normal, and acceleration, that suggests we should apply Newton's Second Law. To apply Newton's Second Law we draw a free-body diagram (FBD) for the car. The car has mass, so it has weight. There is the normal force from the track on the car. By definition, the centripetal acceleration points towards the centre of the track; in this case down.

Applying Newton's Second Law, $F_y = ma_y$, we get $-N - mg = -mv^2/R$. The car loses contact when $N = 0$, so our equation for the speed at which the car leaves the track becomes $v = [gR]^{½} = [9.81×150]^{½} = 38.4$ m/s = 138 km/h. If the car goes faster than this speed, it will remain safely in contact with the track.

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10. The brightest four moons of Jupiter were discovered by Galileo with one of his earliest telescopes. These moons, Io, Europa, Ganymede, and Callisto, are called the Galilean moons in his honour. Some of the available data about these moons are given below.

MOON	**r (km)**	**v**	**T (earthyears)**
Io	4.219×10^5	-	0.004837
Europa	6.712×10^5	-	-
Ganymede	-	-	0.0195884
Callisto	1.853×10^6	-	-

The radii are from the centre of Jupiter to the centre of the moon in question. One earth year has 365 days. From the above data, determine (a) the mass of Jupiter, (b) the period of Europa, (c) the distance between Jupiter and Ganymede, and (d) the speed of Callisto.

Using Kepler's Law, and the data for Io, we can find the mass of Jupiter,

$$M_{Jupiter} = 4\pi^2(R_{Io})^3 / G(T_{Io})^2 .$$

First note that one year is

$$T_{year} = (365\text{ d}) \times (24\text{ h}) / (1\text{ d}) \times (3600\text{ s}) / (1\text{ h}) = 3.154 \times 10^6\text{ s} .$$

So Io's period is

$$T_{Io} = 0.004837\ T_{year} = 1.526 \times 10^4\text{ s}.$$

So the mass of Jupiter is

$$M_{Jupiter} = 4\pi^2(4.219{\times}10^8\text{ m})^3/(6.672{\times}10^{-11}\text{ N-m}^2/\text{kg}^2)(1.526{\times}10^4\text{ s})^2 = 1.9097{\times}10^{27}\text{ kg} .$$

Now that we have the mass of Jupiter, we can find the period of Europa,

$$T_{Europa} = 2\pi(R_{Europa})^{3/2} / [GM_{Jupiter}]^{½} .$$

Using the given values,

$$T_{Europa} = 2\pi(6.712{\times}10^8)^{3/2} / [(6.672{\times}10^{-11})(1.9097{\times}10^{27})]^{½} = 3.061 \times 10^5\text{ s} .$$

In earth years, this is

$$T_{europa} = (3.061 \times 10^5\text{ s}) \times (\text{ 1 year }) / (3.154{\times}10^6\text{ s}) = 0.009706\text{ y} .$$

We find the orbital radius of Ganymede using,

$$R_{Ganymede} = [GM_{Jupiter}(T_{Ganymede})^2 / 4\pi^2]^{1/3} .$$

Ganymede's period is

$$T_{Ganymede} = 0.0195884\ T_{year} = 6.1782 \times 10^4\text{ s}.$$

Thus

$$R_{Ganymede} = [(6.672{\times}10^{-11})(1.9097{\times}10^{27})(6.1782{\times}10^4)^2 / 4\pi^2]^{1/3} = 1.072 \times 10^6\text{ km}.$$

The orbital velocity is given by the formula

$$V_{Callisto} = [GM_{Jupiter} / R_{Callistor}]^{½} .$$

Using the given data, we find

$$V_{Callisto} = [(6.672\times10^{-11})(1.9097\times10^{27}) / (1.853\times10^{6})]^{½} = 8292 \text{ m/s} .$$

Callisto's orbital speed is 2.985×10^4 km/h.

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11. The mass of the planet Mercury is 3.30×10^{23} kg and its radius is 2.439×10^6 m. What would a 65.0-kg person weigh on Mercury? What is the acceleration due to gravity on Mercury.

   Weight is the gravitational attraction of the planet which can be found from The Universal Law of Gravitation,

$$W = GM_{Mercury}M_{person} / R^2 .$$

   Using the given values,

$$W = (6.672\times10^{-11} \text{ N-m}^2/\text{kg}^2)(3.30\times10^{23} \text{ kg})(65 \text{ kg})/(2.439\times10^6 \text{ m})^2 = 241 \text{ N} .$$

   We could have taken a slightly different route to the same answer by first calculating the acceleration due to gravity on Mercury's surface using

$$g_{mercury} = GM_{Mercury} / R^2 .$$

   Using the given values,

$$g_{mercury} = (6.672\times10^{-11} \text{ N-m}^2/\text{kg}^2)(3.30\times10^{23} \text{ kg})/(2.439\times10^6 \text{ m})^2 = 3.70 \text{ m/s}^2 .$$

   The person's weight is then

$$W = M_{person}g_{mercury} = (65.0 \text{ kg})(3.70 \text{ m/s}^2) = 241.0 \text{ kg} .$$

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12. Ted and Alice are mutually attracted to one another in the gravitational sense. If Ted's mass is 80.0 kg and Alice's is 55.0 kg and they are 0.150 m apart, what is the magnitude of the attraction on each? Treat both people as spheres.

   Gravitational attraction is governed by the formula

$$F = GM_1M_2 / R^2 .$$

   Using the given values,

$$F = (6.672\times10^{-11} \text{ N-m}^2/\text{kg}^2)(80 \text{ kg})(55 \text{ kg})/(0.15 \text{ m})^2 = 1.30 \times 10^{-5} \text{ N} .$$

This is a very small amount, so the gravitational force between normal sized object is negligible compared to the objects' weight.

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13. What is the weight of an 80.0-kg cosmonaut on the space station *Mir*, 420 km above the surface of the earth? What is the acceleration due to gravity? Find the mass and radius of the Earth from your text.

    Weight is the gravitational attraction of the planet which can be found from The Universal Law of Gravitation,

    $$W = GM_{Earth}M_{cosmonaut} / R^2 ,$$

    where R is the distance from the centre of the earth to *Mir*.

    The mass of the Earth and its radius may be looked up in many places, including the textbook. Using these values,

    $$W = (6.672\times10^{-11} \text{ N-m}^2/\text{kg}^2)(5.98\times10^{24} \text{ kg})(80 \text{ kg})/(6.378\times10^6\text{m} + 4.20\times10^5\text{m})^2 = 691 \text{ N} .$$

    The acceleration due to gravity can be calculated either from $g(R) = GM_{Earth}/R^2$ or more simply from

    $$g = W/m = 691 \text{ N} / 80.0 \text{ kg} = 8.63 \text{ m/s}^2 .$$

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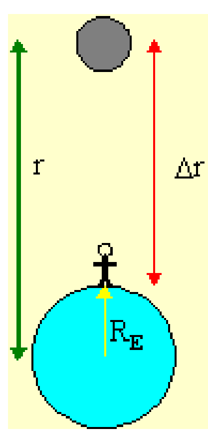
14. Given that the mass of the moon is 7.35×10^{22} kg, that the distance between the centres of the earth and the moon is 3.85×10^8 m, and that the radius of the earth is 6378 km, find the gravitational pull of the moon on a 75-kg person when the moon is directly overhead. Compare (i.e. take the ratio) this to the person's weight.



The force acting on the person is given by the Universal Law of Gravitation,

$$F = GM_M m / (r)^2 ,$$

where $r = r - R_E = 3.85 \times 10^8$ m - 6.378×10^6 m $= 3.78622 \times 10^8$ m. Therefore the gravitational force from the moon is

$$F = (6.672\times10^{-11})(7.35\times10^{22})(75)/(3.78622\times10^{8})^2 = 2.57 \times 10^{-3}\ \text{N} .$$

The person's weight on the other hand is

$$W = mg = (75\ \text{kg})(9.81\ \text{m/s}^2) = 736\ \text{N} .$$

The ratio of the two is

$$F / W = (2.57 \times 10^{-3}) / 736 = 3.5 \times 10^{-6} .$$

So the gravitational effect of even the nearest celestial body, the moon, is negligible comparing to the earth's gravitational pull on objects on its surface.

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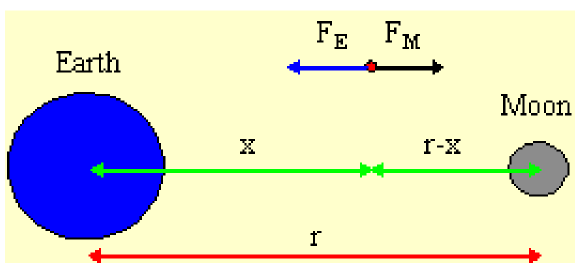
15. The distance between the centres of the earth and the moon is 3.85×10^8 m. The moon has a mass which is only 1.29% that of earth. Where would a satellite have to be placed to feel no net gravitational pull from the earth and the moon?



There are two gravitational forces that an object would feel, F_E and F_M. Since there is no net force, we have

$$F_E - F_M = 0 . \qquad (1)$$

Using the Law of Gravitation, this becomes

$$GM_E m / x^2 = GM_M m / (r-x)^2 .$$

Eliminating common factors yields

$$M_E / x^2 = M_M / (r-x)^2 .$$

Taking the square root of each side, we get

$$[M_E]^{1/2} / x = [M_M]^{1/2} / (r-x) .$$

Cross multiplying and rearranging,

$$r-x = x\,[M_M / M_E]^{1/2} ,$$

or , after we use the fact that $M_M = 0.0129\ M_E$,

$$r-x = (0.11358)x .$$

Solving for x, we find

$$x = r / [\ 1 + 0.11358] = (3.85 \times 10^8\ m)/ 1.11358 = 3.46 \times 10^8\ m.$$

An object would have to be 3.46×10^8 m away from the earth, along a line from the earth to the moon, for the net gravitational pull to be zero.

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